Generalizations of Convergence from $\mathbb{R}$ to $\mathbb{R}^2$

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Sequential convergence is a powerful tool in the field of real analysis. Though its structure persists throughout various metric spaces, students initially understand sequential convergence as it manifests on the real line. Students often do not encounter more generalized forms until advanced analysis courses. As part of multiple teaching experiments, students were given the opportunity to generalize sequential convergence from $\mathbb{R}$ into the $\mathbb{R}^2$. This report will demonstrate various generalizations rooted in reflective abstraction of convergence in $\mathbb{R}$. We will also discuss students generalizing by reduction, reflecting on the utility of distance as a map between spaces.

Key words: generalization, real analysis, convergence, limits, advanced calculus, vectors

Introduction

Convergence is a phenomenon encountered at all levels of mathematical practice. A utility of sequential convergence is its persistent structure throughout metric spaces. Students studying introductory real analysis encounter the convergence of real number sequences and also of continuous functions. These contexts for convergence may be leveraged to facilitate understanding of convergence in more abstract spaces through generalization. These spaces create unique opportunities for students to generalize their understandings in productive ways.

During the selection interviews for multiple teaching experiments, students were given an opportunity to generalize convergence of real numbers to the convergence of real vectors in two dimensions. Their work revealed multiple instances of generalization rooted in abstraction of real number convergence. In this report I seek to answer the following research question: How do undergraduate students leverage convergence of real numbers when defining convergence in more abstract spaces?

Literature Review

Student Understanding of Convergence

While student understanding of limits and convergence has been thoroughly investigated, there is still much to learn about how students understand convergence beyond introductory contexts. Early studies investigated student initial understanding of limits, problems that may result from students' initial understandings, and intuitions behind the limit concept (Bezuidenhout, 2001; Cornu, 1991; Cottrill, Dubinsky, Nichols, Schwingendorf, Thomas, & Vidakovic, 1996, Davis & Vinner, 1986; Oehrtman, 2003/2009; Roh, 2008/2009; Tall, 1992; Tall & Vinner, 1981; Williams, 1991). Many of these studies focused primarily on student understanding of informal limiting processes, leaving room for investigations of formal limiting processes.

In 2011, Swinyard began investigating students' formal understanding of limits via a teaching experiment in which two students reinvented the formal definition of a limit. Useful constructs for describing student formal understanding of limits emerged from this experiment. Studies that followed expanded on his work, proposing strategies for fostering useful understanding of limits and also reinventing the formal definition for sequential convergence (Swinyard & Larsen, 2012; Oehrtman, Swinyard, & Martin, 2014).

Other studies have also examined student understanding of formal limiting processes. Adiredja investigated how students make sense of the relationship between the multiple limit-controlling variables (Adiredja, 2013/2015; Adiredja & James, 2013/2014). Also, Roh and colleagues (Dawkins & Roh, 2016; Roh, 2009; Roh & Lee, 2016) implemented interventions such as the "Mayan Activity" and the "ϵ -strip activity" designed to illuminate the logical structure of formal convergence. Finally, Reed (2017) examined a student's understanding of the logical structure of point-wise convergence for functions. These studies examine the nuances that accompany the logical statement of convergence.

Generalization

While the activity of generalizing has been investigated in many contexts, such studies have not yet examined students generalizing formal mathematics. Generalization has been deemed a relevant mathematical activity both by researchers (Amit & Klass, 2005; Lannin, 2005; Pierce, 1902; Vygotsky, 1986) and educators (Council of Chief State School Officers, 2010). Indeed, generalization has been thoroughly investigated in algebraic and other elementary contexts (Amit & Neria, 2008; Becker & Rivera, 2006; Carpenter, Franke, & Levi, 2003; Ellis, 2007a/2007b/2011; Radford, 2006/2008; Rivera, 2010; Rivera & Becker, 2007/ 2008).

More recent investigations have begun to explore student generalizations at the undergraduate level. Researchers have studied student generalizations in both single and multi-variable calculus (Dorko, 2016; Dorko and Lockwood, 2016; Dorko & Weber, 2014; Fisher, 2008; Kabael, 2011; Jones and Dorko, 2015) as well as combinatorics (Lockwood and Reed, 2016). For instance, Jones and Dorko (2015) considered different ways in which the multivariable integral is understood as a generalization of notions that students held for single variable integrals, such as generalizing from an "area under the curve" model in single variable calculus. While these studies investigate the nature of generalizing activity in various advanced contexts, the research so far has not investigated generalization of formal mathematics.

This report contributes both to the literature on convergence and to the literature on generalization by observing students generalize the concept of convergence in a formal context.

Theoretical Perspectives

Generalization

We wish to characterize the activities students engage in while generalizing. To do this, we consider student activity according to Ellis' (2007a) taxonomy of students' generalizing activity. This examines generalization from an actor-oriented perspective (Lobato, 2003).

Ellis described three broad categories of generalizing activity in which students engage: *relating, searching* and *extending*. In *relating*, "a student creates a relation or makes a connection between two (or more) situations, problems, ideas, or objects" (p. 235). Generalizing activity can manifest as an organizing of similar situations which then become the source material for further generalizations.

The next activity students engage in is to search for a pattern or relationship. This is where students perform "the same repeated action in an attempt to determine if an element of similarity will emerge" (p.238). The distinction here is that the student repeats an activity to uncover some regularity.

Finally, students engage in *extending*. This occurs when a student "not only notices a pattern or relationship of similarity, but then expands that pattern or relationship into a more general structure" (p.241). This extension can be done in multiple ways that expand the source material to new abstraction. *Extending* moves beyond the observance of relationships or patterns, and involves the creation of new mathematical objects that reflect the source of the generalization in some way. These three categories provide us language with which to observe and discuss the generalizing activity of students in any mathematical setting.

Abstraction

We find Piaget's notion of *abstraction* (Piaget, 2001; Glasersfeld, 1995) to be complementary to studying student generalization. Indeed, through *abstraction* Piaget describes the cognitive mechanisms through which activity is reorganized and extended. Specifically, we are concerned with facilitating *reflective abstraction* (Glasersfeld, 1995). In *reflective abstraction*, an operation (mental activity) "developed on one level is abstracted from that level of operating and applied to a higher one" (Glasersfeld, 1995, p. 104). Indeed, *reflective abstraction* accompanies generalization as it can describe mathematical activity being organized at higher levels of thought. *Reflective abstraction* is characterized in two parts. The first is a réflexion, or reflection, of the operations from their original context (p. 104). This indeed highlights the importance of salient activity from which to abstract. The second part of reflective abstraction is a réfléchissement, or projection, of the borrowed operation to a higher level of thought (p. 104).

Thus we see *reflective abstraction* involving the borrowing of activity to then be applied at higher levels of thought. In mathematical contexts, this indeed can be used to characterize the generalization of operations. Using *reflective abstraction* as an underpinning for generalizing activity allows us to use mathematical activity as a direct source of generalization. This indeed will be useful in describing the generalizations of students as they engage in extending their mathematical understandings.

Mathematical Discussion

Convergence is a generalizable concept that obeys the same structure in various real spaces. Convergence of real number sequences and other seemingly more complex objects, such as uniformly convergent function sequences, are indeed the same because of the metric structure associated with each space. Consider the formal definition for convergence of real numbers: A sequence $\{x_n\}$ of real numbers converges to a real number x if $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$ such that $\forall n \geq N$, we have $|x_n - x| < \epsilon$. The mathematical structure of such convergence stems from the definition of convergence within any general metric space: A sequence $\{x_n\}$ in a metric space (M, δ) with a metric δ converges to x if $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$ such that $\forall n \geq N$, we have $\delta(x_n, x) < \epsilon$. Such is the case for real vectors in $\mathbb{R}^2$. Indeed, the only alteration to make in each metric space is the notion of distance. On the real line, distance is measured using the absolute value norm. While many equivalent metrics may be applied in the plane, perhaps the most natural is the metric given by the Euclidean distance. Indeed, when the distance between vectors is measured using the Euclidean distance, the convergence of a sequence of real vectors may be characterized as follows: A sequence of vectors $\{\vec{x_n}\}$ in $\mathbb{R}^2$ converges to a vector $\vec{x} \in \mathbb{R}^2$ if $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$ such that $\forall n \geq N$, we have $\sqrt{(x_n^1 - x^1)^2 + (x_n^2 - x^2)^2} < \epsilon$. Note that in this notation x^k represents the k-th component of the vector $\vec{x}$.

Similarly, the characterization of Cauchy sequences is uniform throughout metric spaces: A sequence $\{x_n\}$ in the metric space (M, δ) with metric δ is Cauchy if $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$ such that $\forall n, m \geq N$, we have $\delta(x_n, x_m) < \epsilon$. This allows for Cauchy sequences to be characterized on the real line by: A sequence $\{x_n\}$ of real numbers is Cauchy if $\forall\epsilon>0$, $\exists N \in \mathbb{N}$ such that $\forall n, m \geq N$, we have $\delta(x_n, x_m)<\epsilon$, and on the real plane by: A sequence of vectors $\{\vec{x_n}\}$ in $\mathbb{R}^2$ is Cauchy if $\forall\epsilon>0$, $\exists N \in \mathbb{N}$ such that $\forall n, m \geq N$, we have $\sqrt{(x_n^1 - x_m^1)^2 + (x_n^2 - x_m^2)^2)}<\epsilon$.

Methods

This study reports on the selection interviews and the first sessions of multiple teaching experiments (Steffe and Thompson, 2000) being conducted for my dissertation. Two of the teaching experiments conducted were for pilot purposes, and consisted of six hour-long sessions each with one student. The other two teaching experiments consisted of ten 90-minute sessions, one with a pair of students and one with an individual student. The students reported on were each mathematics majors at a large university. Each were recruited out of the advanced calculus sequence, and had finished at least one course in the sequence before the interviews were conducted.

The sessions were semi-structured and task-based (Hunting, 1997) so that student activity could be observed and understanding could be inferred. The goal of these episodes was to facilitate reflection on real-number convergence and the characterization of the absolute value as a measurement of distance. Thus, the interviews began with discussions of real number convergence and distance measurement on $\mathbb{R}$ and $\mathbb{R}^2$. The students were prompted to write down and explain their definitions of real number convergence, and then to demonstrate that a specific sequence converges. Once details were discussed, and distance measurement was thoroughly discussed, the discussion turned to convergence of vectors. The students were prompted to characterize convergence of a sequence of vectors, and similarly demonstrate that a specific sequence of vectors converges. Any further discussion negotiated nuances of their multiple characterizations of convergence.

The interviews were video recorded, and the records have been reviewed multiple times looking for episodes of student generalization to be analyzed using Ellis' (2007a) framework and Piaget's (2001) construct of reflective abstraction.

Results

Convergence on $\mathbb{R}$

Each interview began with a review of the known distance and convergence concepts on the real line. I will describe an understanding commonly held by the students that was relevant for their generalizations of convergence.

When prompted to characterize convergence of a sequence of real numbers, each student gave the standard ϵ-N definition described above in the mathematics section. The discussion that followed allowed the students to further explain their understanding of the characterization. For instance, Kyle said the following while describing distance measurement via the absolute value function while characterizing a Cauchy-convergent sequence:

> Well we end up measuring the distance between two subsequent points. That's what we're doing here-we're saying that the absolute value of $a_n - a_m$ is the distance between these two points in the sequence. So you're trying to measure, as your n is getting arbitrarily large, what's happening between this point in the sequence [a_n] and this point in the sequence [a_m].

The absolute value as a distance measurement on the real line similarly emerged in all of the interviews. For instance, using $1/n$ as an example, Jake said the following while explaining why we take the absolute value to characterize Cauchy-convergent sequences:

> If we had this instead of $1/n$ be $-1/n$ ... we're just concerned with the width between the numbers not where they are relative to the x-axis ... 'cause we're always concerned with the relative distance of the two. Not if it's, you know, below of above the x-axis. The absolute value takes care of that.

Similar discussions of real number convergence were had with all students in the study. Each student displayed a sophisticated understanding of sequential convergence on the real line. These discussions then influenced the students' generalizations of convergence to $\mathbb{R}^2$

Convergence on $\mathbb{R}^2$

Following discussions of convergence on the real line and characterizations of distance in $\mathbb{R}$ and $\mathbb{R}^2$, the students were prompted to develop a characterization of vector convergence. Two distinct generalizations emerged from the students' characterizations of vector convergence. Both generalizations result from reflections on real number convergence.

The first generalization involves considering vectors on the real plane component-wise and isolating real number sequences in each component. This generalization manifested differently among the students, each instance demonstrating unique understandings. Laura and Kyle considered vectors in terms of their components separately, and described a sequence of vectors as a pair of real-number sequences. This allowed for convergence of the sequences to rely on convergence of the components. Below, Laura described the sequence of vectors converging in the following manner:

> If this [a sequence of vectors $\{\vec{v_n}\}$] converges, then that means your x-component has to converge and your y-component has to converge. Which is realistically seeing if - two independent sequences converges - you have some sequence $\{x_n\}$ and some sequence $\{y_n\}$ that make up your vector, then it's basically like doing the convergence thing twice but you have to fit it for both x and y.

Here Laura extends convergence of real numbers to a vector setting by making note that a sequence of vectors forms two sequences of real numbers, and asserts that controlling the convergence of each component will result in a convergent sequence of vectors. Note that here, Laura is using the known structure for convergence of real numbers.

Jerry, Jake and Christina produced similar component-wise characterizations, however their approaches differed subtly from Kyle and Laura. Specifically, Jake and Christina attempted to bound the sequence of difference vectors by an "error vector". When generalizing convergence to two dimensions, Christina initially wrote out again the definition of real number convergence, and used the same notation as real-number convergence, while noting the caveat that in this case $|A_n - A|$ denoted a distance between vectors. Moreover, she described making the distance vector "smaller" than some error vector (a, b):

> So, this [$A_n^1 - A^1$] is describing the horizontal distance that will be traveled, and then this [$A_n^2 - A^2$] is describing the vertical distance that will be traveled. And the whole entire thing describes a vector that would create that translation, and it's going to be - I guess less than would actually be smaller than - the ϵ vector.

Christina went on to describe that in this context “less than” does not necessarily mean an ordering, but instead referred to the sizes of the vectors. From this, she reduced convergence to comparing the components of the difference vector $\{A_n - A\}$ and the ”error vector”. She then reduced the vector comparison to a component-wise comparison and arrived at a similar characterization as above. Similarly, Jake wrote a characterization for Cauchy-convergence of vectors that was identical to Christina’s up to notation. Motivated by the behavior of a sequence when the y-axis is constant, Jake said the following:

> And you do the same with the x-axis. You get an analogous statement with the x-components [where keeping the x-components constant yields a real number sequence in the y-components]. But we have the difference between the x-components and the y-components both decrease below some arbitrary ϵ. Otherwise you could have convergence with respect to the y-axis but having it oscillate back and forth in the x, or increase without bound on the x or vice versa.

So for Jake, while the problem of convergence was to capture varying vectors, the variation could be simultaneously handled in both components.

Thus we see these students were able to reduce the problem of vector convergence into a form that they are familiar with, namely convergence on the real line. In this case, they reflected that they could iterate real number convergence a finite amount of times (in this case twice), and that convergence of each component implied convergence of the vector as a whole. Mathematically, these characterizations are interesting because multiple geometries can result from considering the component-wise absolute value distances, including the ℓ_1 and ℓ_∞ distances.

The final generalization we will discuss differs from the generalizations above in that it involves reflection on the role of measuring distance in convergence, rather than taking advantage of the repeatability of real number convergence. It involves reflection on structures that are more consistent with a general metric. After being challenged to find a characterization of convergence that involves a single calculation rather than multiple calculations, Jerry and Christina together used the Euclidean distance formula to create a sequence of the distances between vectors in the sequence and the convergent vector. These distances formed a sequence of real numbers that would converge to 0. After formalizing the convergence of the sequence of distances, Jerry made the following statement:

> I like this ‘cause it seems like we reduced the problem to something that was like, that we already know, so I feel like this is on the right track So we now have this number that we can check for every single vector in our sequence and that generates a sequence of real numbers which we already know the rules of convergence for. And that’s something we can check.

Jake constructed a similar generalization of a Cauchy-convergent sequence of vectors. These generalizations allow for direct comparison between vectors involved in the convergence process via a distance calculation. As the students indicated, this calculation allows for the convergence to be stated in terms of a varying set of real numbers, namely the sequence of distances between the elements of the set of vectors converging. This transforms the problem into one of a known operation, namely the convergence of real numbers. This, in fact, is the logical structure of convergence in abstract metric spaces. In contrast, the students’ first generalization is indeed also generalizable, but only within finite dimensional vector spaces, as the requirement of checking convergence in each component is only possible a finite number of times.

Discussion and Conclusion

Reflective Abstraction

Within these episodes we see multiple instances of reflective abstraction when generalizing sequential convergence from the real line to $\mathbb{R}^2$. While both generalizations involve reflection on the structure of convergence along the real line, the réfléchissement of such reflection is manifest in two qualitatively different ways. I infer that the component-wise construction involves abstractly projecting the action of taking a limit on the real line. In observing that vectors can be expressed via components of real numbers, and formulating the sequences of vectors to reflect real number sequences, the students project convergence of real number sequences to two simultaneous iterations of real number sequences that converge in conjunction with one another. This involves reflection on the process of real number convergence, and then projection of this to each component in the vector structure.

Generalizing through reduction

While the second generalization also may be characterized via reflective abstraction, it also reveals a new form of generalizing by extending. Within this episode we also see instances of students generalizing a more abstract phenomenon by reducing aspects of the problem to a known setting. Jerry and Christina, as well as Jake, utilized the Euclidean distance formula as a map from $\mathbb{R}^2$ to $\mathbb{R}$ that reduced the problem from one that involved 2-dimensional geometry back to the convergence of real numbers. Thus, the students manipulated the structure of the problem at hand to match a familiar structure. The use of a mapping in this way is entirely consistent with productive activity in multiple areas of mathematics. As an example, the integral can be similarly used to reduce problems of functional variation to problems of varying real numbers via special limiting processes. These are interesting instances of generalization, as they involve coordinating of an abstract process via simplification. This is indeed reminiscent of Jerry's final comment. Moreover, while Jerry and Christina were challenged to perform a single calculation, their characterization of the problem in terms of a sequence of real-number distances involved constructing an explicit relationship between the vectors that varied in the sequence and the known structure of $\mathbb{R}$.

Conclusions

In this report we see two distinct generalizations rooted in sophisticated understandings of real number convergence. By reflecting on convergence as an activity, the students generated two generalizations unique from each other mathematically and cognitively. The generalization characterizing vector sequences as pairs of real numbers reflectively abstracts the repeatable operation of checking real number convergence. Further, the generalization utilizing the Euclidean distance involves reducing the more abstract mathematical phenomena of vector convergence to the simpler and more familiar setting for convergence, that of the real numbers. These findings begin to illuminate the nature of student thought an generalization in more formal mathematical settings. Specifically, we see students attending to natural relationships in real space to facilitate meaningful generalizations of known analytical phenomena. Further research will investigate student generalization in more abstract spaces.

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